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Report TW 91

A faculty series for the incomplete gamma function and the related error functions

by

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1. Introduction

The incomplete gamma function, a particular case of which frequently occurs in heat-problems in the form

$$Ei(x) = \int_x^{\infty} e^{-t} t^{-1} dt,$$

is generally defined by

(1.1) $$\Gamma_{\alpha}(z) = \int_z^{\infty} e^{-t} t^{\alpha -1} dt,$$

where z is a complex variable and α is an arbitrary real parameter. Using standard methods this functions can be expanded in a power series,

(1.2) $$\Gamma(\alpha) - z^{\alpha} \sum_{n=0}^{\infty} \frac{(-1)^n z^n}{(n+\alpha).n!} \quad (\alpha \neq 0,-1,-2,-3,\ldots),$$

or in an asymptotic series

(1.3) $$z^{\alpha -1} e^{-z} \sum_{n=0}^{\infty} \binom{\alpha -1}{n} \frac{n!}{z^n} \quad (|\arg z| < \pi).$$

For numerical purposes the power series is useful only for small values of $|z|$. The asymptotic series on the other hand can be used only for relatively large values of $|z|$.
In this report a faculty series is derived in which to a certain extent the advantages of the two previous expansions are combined.
The method is briefly as follows. The right-hand side of (1.1) is transformed into the form

$$\int_0^1 (1-v)^{\frac{r}{\mu} -1} F(v) dv,$$

where $r = |z|$ and μ is an arbitrary real positive parameter, which will be used later on to improve the speed of convergence. Then expansion of F(v) in powers of v leads to an expansion in a series of beta-functions, viz.

(1.4) $$\sum_{n=0}^{\infty} c_n B(\frac{r}{\mu}, n+1),$$

where the coefficients c_n satisfy a recurrent relation which will be derived in section 4. From (1.4) the desired faculty series

$$(1.5)\qquad \Gamma_\alpha(z) = \frac{z^\alpha e^{-z}}{\mu \exp(i\varphi)} \sum_{n=0}^{\infty} \frac{c_n \mu^{n+1} n!}{r(r+\mu)\ldots(r+n\mu)}$$

follows at once.

It will be shown that for appropriate values of the free parameter μ this series converges over the whole complex z-plane except the negative real axis. The influence of μ upon the speed of convergence is shown in table 2, at the end of this report. It will also be seen that the series converges rapidly for $|z| \gg 1$ and that for relatively large $|z|$ it gives a better relative accuracy than the power series and the asymptotic series (see table 1 and 3). The relative accuracy is computed with

$$\text{rel.acc.} = \left| \frac{a_n}{S_n} \right| \qquad (n=0,1,2,\ldots),$$

where a_n is the last term of the n^{th} partial sum S_n. In section 5 we consider the important special cases $\alpha=0$ and $\alpha=0.5$ where the incomplete gamma function reduces to the exponential integral and the error function. The result of computation of ten terms is compared with the values, given in the tables [4] - [6]. They show an increasing agreement with increasing value of $|z|$. If cancellation occurs at all (see table 2, z=1) it does not influence the accuracy in a significant way, since it only occurs in relatively small terms (see table 2, where cancellation only occurs in the limiting case z=1 and only after 9 terms). In the use of the power series cancellation is a very troubling fact.

Calculations were carried out with the X-1 computer of the Mathematisch Centrum with the aid of ALGOL-60 programmes.

This investigation was carried out under supervision of Prof.dr. H.A. Lauwerier.

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2. The faculty series

On making the substitution t=zs and s=w+1 in the right-hand side of (1.1) we find

$$\Gamma_\alpha(z) = e^{-z}z^\alpha \int_0^\infty e^{-zw}(w+1)^{\alpha-1}dw, \tag{2.1}$$

where the path of integration is such that arg w = arg t - arg z. If the integration in (1.1) is carried out as shown in fig.1a it can easily be seen that the path of integration in (2.1) must be as shown in fig.1b; this path of course may be taken as a straight one by transforming it without passing singularities.

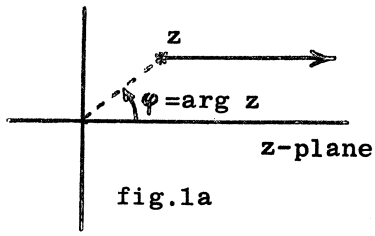

fig.1a

Now we put $z=r\exp(i\varphi)$ and $w=u(\mu\exp(i\varphi))^{-1}$, where arg u=0 and μ is a real positive parameter.

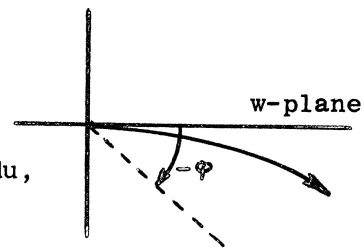

fig.1b

Then (2.1) becomes

$$\frac{z^\alpha e^{-z}}{\mu\exp(i\varphi)} \int_0^\infty \exp(-\frac{ur}{\mu}) \left\{1+\frac{u}{\mu\exp(i\varphi)}\right\}^{\alpha-1} du,$$

or by substituting $e^{-u}=1-v$

$$\Gamma_\alpha(z) = \frac{z^\alpha e^{-z}}{\mu\exp(i\varphi)} \int_0^1 (1-v)^{\frac{r}{\mu}-1} F(v)dv, \tag{2.2}$$

where

$$F(v) = \left\{1-\frac{\ln(1-v)}{\mu\exp(i\varphi)}\right\}^{\alpha-1}. \tag{2.3}$$

The formal expansion of F(v) in a power series may be obtained by using

$$-\ln(1-v) = v + \frac{v^2}{2} + \frac{v^3}{3} + \dots$$

and the binomial theorem, which gives

(2.4) $$F(v) = \left\{1 + \frac{v + \frac{v^2}{2} + \frac{v^3}{3} + \dots}{\mu \exp(i\varphi)}\right\}^{\alpha-1} = \sum_{n=0}^{\infty} c_n v^n,$$

in which $c_0=1$, $c_1 = \frac{\alpha-1}{\mu\exp(i\varphi)}$, $c_2 = \left\{\frac{\alpha-1}{2\mu\exp(i\varphi)} + \frac{(\alpha-1)(\alpha-2)}{2!\,\mu^2\exp(i\varphi)}\right\}$, etc.

A recurrent relation for the coefficients c_n is derived in section 4. Then (2.2) becomes

$$\Gamma_\alpha(z) = \frac{z^\alpha e^{-z}}{\mu\exp(i\varphi)} \int_0^1 (1-v)^{\frac{r}{\mu}-1} \sum_{n=0}^{\infty} c_n v^n \, dv,$$

so that by interchanging summation and integration we obtain, at least formally,

(2.5) $$\Gamma_\alpha(z) = \frac{z^\alpha e^{-z}}{\mu\exp(i\varphi)} \sum_{n=0}^{\infty} c_n B\left(\frac{r}{\mu}, n+1\right).$$

Using the well-known relation

$$B\left(\frac{r}{\mu}, n+1\right) = \frac{\Gamma\left(\frac{r}{\mu}\right)\Gamma(n+1)}{\Gamma\left(\frac{r}{\mu}+n+1\right)} = \frac{n!}{\frac{r}{\mu}\left(\frac{r}{\mu}+1\right)\dots\left(\frac{r}{\mu}+n\right)}$$

we finally obtain *)

(2.6) $$\Gamma_\alpha(z) = \frac{z^\alpha e^{-z}}{\mu\exp(i\varphi)} \sum_{n=0}^{\infty} \frac{c_n \mu^{n+1} n!}{r(r+\mu)\dots(r+n\mu)}.$$

The convergence of this series will be investigated in the following section.

3. Convergence

The transition of (2.2) to (2.5) is permitted if the power series expansion of F(v) converges for $|v| \leq 1$. For this it is a sufficient condition that the singularities of F(v) are outside the unit circle. Therefore we first investigate the possible singularities of F(v).

It is easily seen that, excluding the trivial cases $\alpha=1,2,\dots$, where F (z) is an elementary function, the singularities of F(v) are given by

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*) For an other derivation in the special case that z is a real variable, see [3] .

v=1 and the relation

$$\frac{\ln(1-v)}{\mu \exp(i\varphi)} = 1. \tag{3.1}$$

The singularities are in general of the branch-point type. The position of the singularity (3.1) can be found from

$$v = 1 - \exp(\mu e^{i\varphi}) = x + iy \tag{3.2}$$

and hence

$$\begin{cases} x = 1 - \exp(\mu \cos\varphi) \cdot \cos(\mu \sin\varphi), \\ y = -\exp(\mu \cos\varphi) \cdot \sin(\mu \sin\varphi). \end{cases} \tag{3.3}$$

Now the condition imposed on F(v) demands $x^2+y^2 \gtrsim 1$, so using (3.3) we find the condition

$$\exp(\mu \cos\varphi) \gtrsim 2 \cos(\mu \sin\varphi), \tag{3.4}$$

the interpretation of which is that for any given value of φ the parameter μ should satisfy a certain inequality. This relation between φ and μ is explained geometrically in figs.3 and 4.

In fig.3 for points (μ, φ) within the shaded regions the condition (3.4) ceases to hold.

Writing $\mu \cos\varphi = \xi$ and $\mu \sin\varphi = \eta$ we see that μ and φ are the polar coordinates in the $\xi - \eta$ plane (fig.3). Obviously μ must be taken large when φ approaches $\pm\pi$. This is shown explicitly in fig.4.

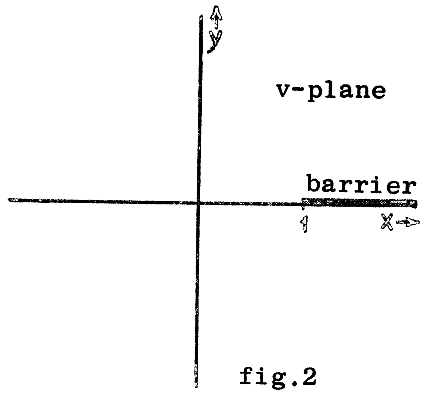

fig.2

From (2.3) it is possible to find an asymptotic expression of c_n for large n, using the expression

$$c_n = \frac{1}{2\pi i} \oint \frac{F(v)}{v^{n+1}} dv.$$

The path of integration is blown up until there remain a path along the barrier $[1,\infty]$ and a contribution from the singularity (3.2). Since the latter singularity is outside the unit circle the asymptotic behaviour of c_n is determined only by the branch-point at v=1. Here we shall merely

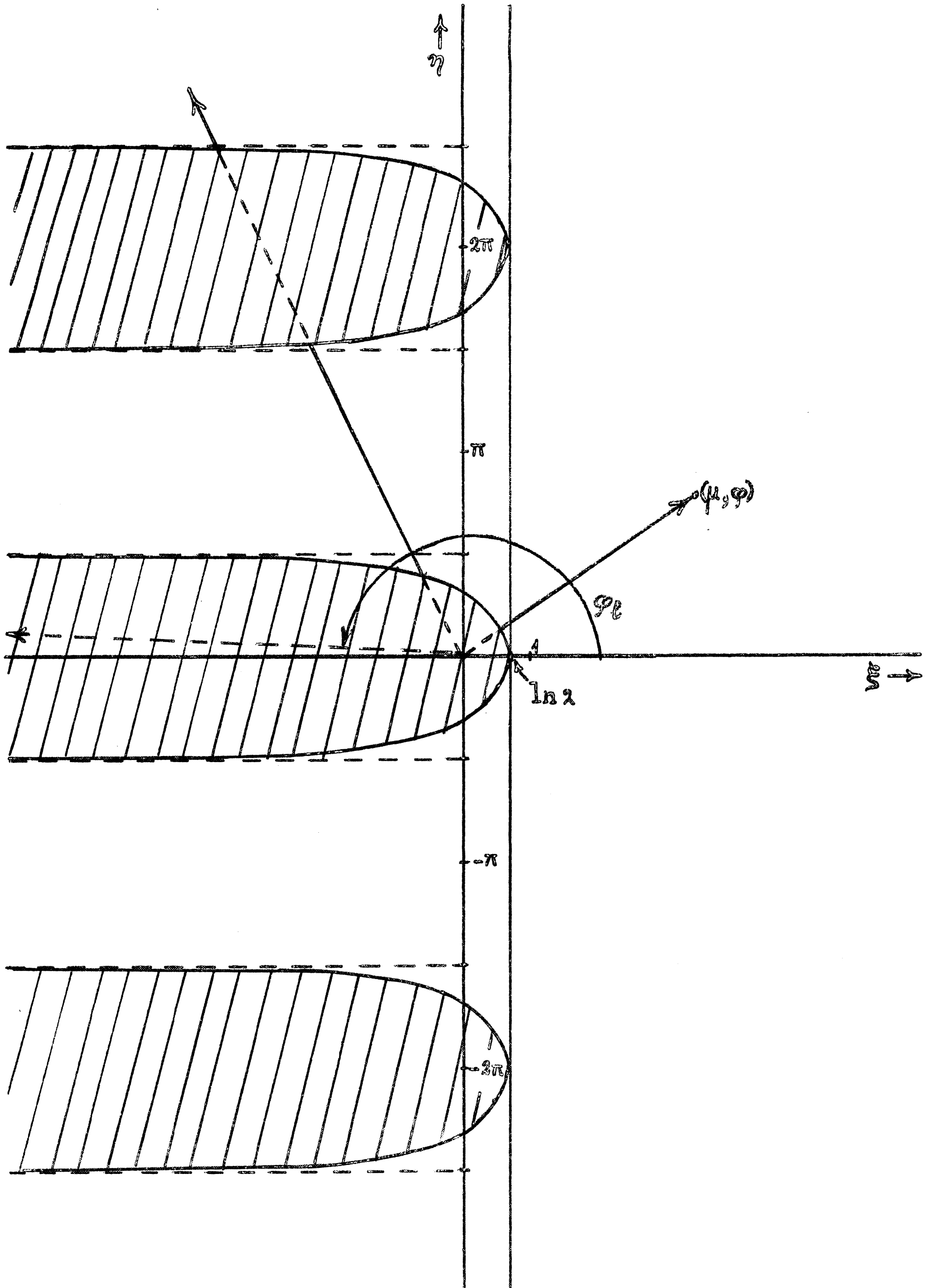

fig.3
curves $e^{\xi} = 2\cos\eta$

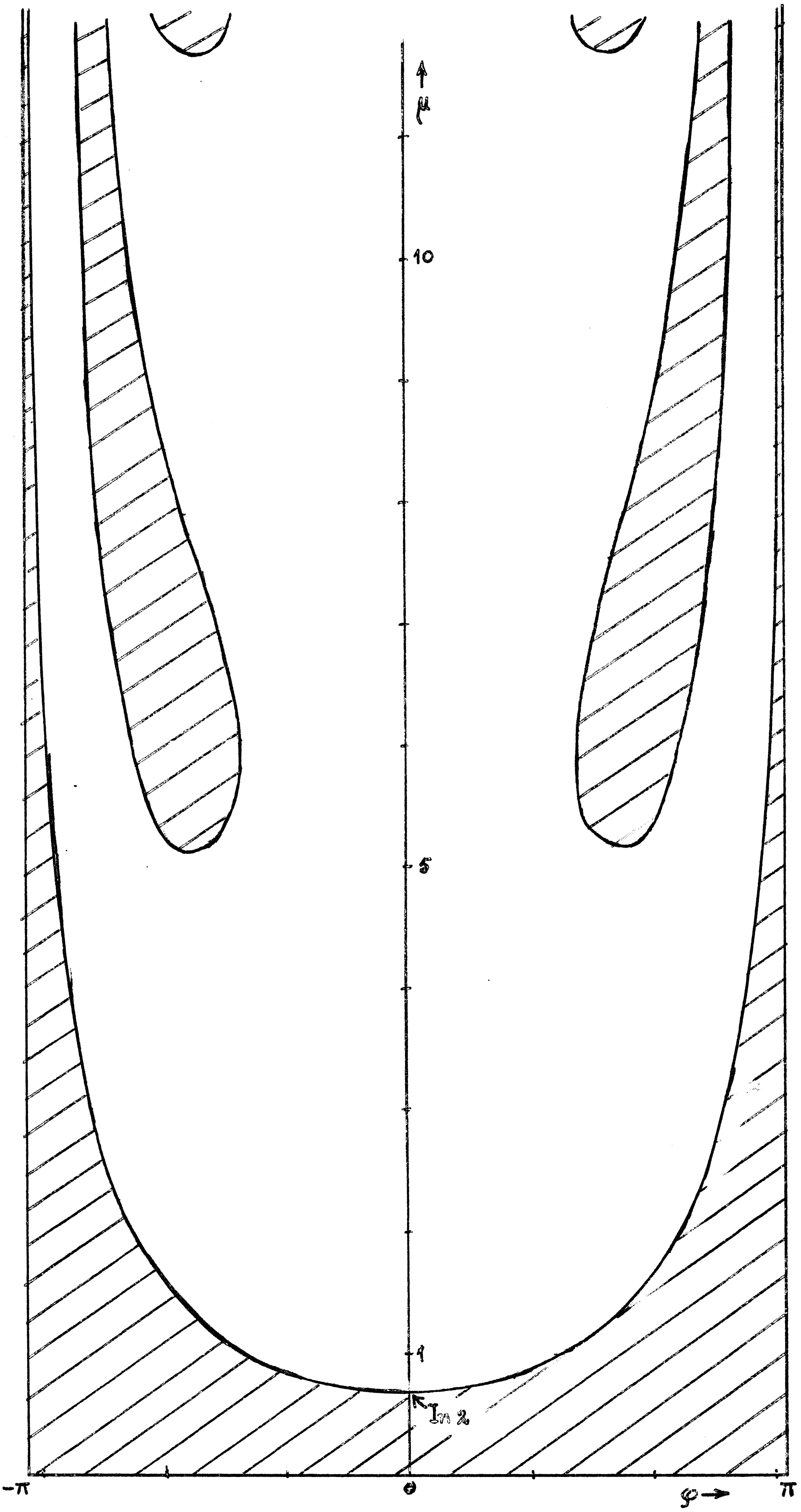

fig.4

$\mu - \varphi$ diagram

state the result

$$c_n \approx \frac{1}{n \log^2 n} .$$

4. A recurrent relation between the coefficients c_n

From (2.3) a recurrent relation of the coefficients of (2.4) can be derived in the following way. Putting $(\mu \exp(i\varphi))^{-1} = \lambda$ we have

$$(4.1) \qquad \begin{cases} F(v) = \left\{1 - \lambda \ln(1-v)\right\}^{\alpha-1}, \\ F'(v) = \lambda(\alpha-1) \dfrac{\left\{1-\lambda \ln(1-v)\right\}^{\alpha-1}}{(1-v)\left\{1-\lambda \ln(1-v)\right\}}, \end{cases}$$

and consequently

$$(4.2) \qquad \lambda(\alpha-1) . F(v) = (1-v)\left\{1-\lambda \ln(1-v)\right\} F'(v) .$$

Since

$$(4.3) \qquad \begin{cases} F(v) = \sum\limits_{n=0}^{\infty} c_n v^n , \\ F'(v) = \sum\limits_{n=1}^{\infty} n\, c_n v^{n-1} , \end{cases}$$

and

$$(1-v)\left\{1-\lambda \ln(1-v)\right\} = (1-v)\left\{1+\lambda v + \frac{\lambda v^2}{2} + \frac{\lambda v^3}{3} + \dots\right\} =$$

$$= 1+(\lambda-1)v + \lambda \sum_{n=1}^{\infty} \frac{v^{n+1}}{n(n+1)} ,$$

we have the identity

$$(4.5) \qquad \lambda(\alpha-1) \sum_{n=0}^{\infty} c_n v^n = \left\{1+(\lambda-1)v - \lambda \sum_{n=1}^{\infty} \frac{v^{n+1}}{n(n+1)}\right\} \cdot \sum_{n=1}^{\infty} n\, c_n v^{n-1} .$$

Equating the coefficients of both sides we obtain

$$\lambda(\alpha-1)c_n = (n+1)c_{n+1} + (\lambda-1)n\, c_n - \lambda \sum_{j=1}^{n-1} \frac{j\, c_j}{(n-j)(n-j+1)} ,$$

or explicitly, inserting the value of λ again,

$$(4.6) \qquad c_{n+1} = \frac{1}{(n+1)\ \mu \exp(i\varphi)} \left[\sum_{j=1}^{n-1} \frac{jc_j}{(n-j)(n-j+1)} + \left\{ \alpha - n - 1 + n\ \mu\ \exp(i\varphi) \right\} c_n \right]$$

for $n \geq 0$ while in section 2 it was found that $c_o=1$.
If $\varphi \neq 0$ if desired this complex expression may be transformed into two coupled real recurrent relations.

5. Numerical treatment

In order to demonstrate the use of the faculty series derived above we consider the special case of the exponential integral and the error function which are extensively tabulated by various authors.
In each case a few values of $\Gamma_\alpha(z)$ were calculated by using the faculty series which was truncated after the 10^{th} term and compared to the corresponding values as given in the tables.

I. First we consider the exponential integral

$$E_n(x) = \int_1^\infty e^{-xu} u^{-n} du,$$

(where x is real) which is tabulated by Placzek [4] .
Obviously

$$(5.1) \qquad E_n(x) = x^{1-n} \Gamma_{1-n}(x).$$

Taking n=1, so $\alpha=0$ we have

$$(5.2) \qquad E_1(x) = \Gamma_o(x).$$

Giving the free parameter μ the value 1 we found

x	Table of $E_1(x)$	calculated with fac.ser.
1	0.2193839	0.22048 43581 96
5	0.114830 (-2)	0.11482 98935 19 (-2)
9	0.12447 (-4)	0.12447 35472 30 (-4)

II. Further we consider the probability integral as given in the table [6b]

$$Q(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty \exp(-\frac{u^2}{2})du,$$

where x is real.

Simple calculations show

$$Q(x) = \frac{1}{\sqrt{\pi}} \Gamma_{\frac{1}{2}}(\frac{x^2}{2}).$$

The following results with α =0.5 were obtained

μ	x	Table of prob.fu.II			calculated			
1	18	0.19731	75	(-8)	0.19731	75291	17	(-8)
1	32	0.12441	92	(-14)	0.12441	92114	86	(-14)
1	50	0.15239	71	(-22)	0.15239	70604	83	(-22)
π	18				0.19731	80024	81	(-8)
π	32				0.12441	92179	10	(-14)
π	50				0.15239	70607	48	(-22)

Clearly a larger value of μ decreases the speed of convergence a little (see table 1 where all computed terms are given). When z is complex this disadvantage is neutralized by the applicability to any value of arg z (if not $\pm\pi$).

III. From [6a] the following error function had been taken

$$H(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-u^2)du, \tag{5.5}$$

where x is real.

It is easily seen that after simple substitutions

$$H(x) = 1 - \frac{1}{\sqrt{\pi}} \Gamma_{\frac{1}{2}}(x^2). \tag{5.6}$$

Taking α =0.5 and μ=1 we found

x	Table of prob.fu.I			calculated		
1	0.84270	07929	49715	0.84185	39250	51
3	0.99532	22650	18953	0.99532	21812	30
5	0.99999	99999	98463	0.99999	99999	98

IV. This last example concerns the case that z is complex. We took the function

$$(5.7) \qquad W(z) = \exp(-z^2) \int_0^z \exp(t^2)\,dt$$

from [5] . Substituting $t=s^2$ in (1.1) and taking α =0.5 a simple calculation shows

$$\Gamma_{\frac{1}{2}}(z) = 2\left[\int_0^{\infty} \exp(-s^2)\,ds - \int_0^{\sqrt{z}} \exp(-s^2)\,ds\right] =$$

$$= \sqrt{\pi} - 2\int_0^{\sqrt{z}} \exp(-s^2)\,ds,$$

and so

$$\Gamma_{\frac{1}{2}}(z^2) = \sqrt{\pi} - \int_0^{z} \exp(-s^2)\,ds.$$

Substitution of s=it (t complex) gives

$$\Gamma_{\frac{1}{2}}(z^2) = \sqrt{\pi} - 2i\int_0^{-iz} \exp(t^2)\,dt,$$

and if z is replaced by -iz we obtain

$$\Gamma_{\frac{1}{2}}(-z^2) = \sqrt{\pi} - 2i\int_0^{z} \exp(t^2)\,dt,$$

so that

$$W(z) = \frac{i}{2}\exp(-z^2)\left\{\Gamma_{\frac{1}{2}}(-z^2) - \sqrt{\pi}\right\},$$

where $z=r\exp(i\varphi)$.

Computation of 10 terms with $\mu=3$, $r=|z|=5$ gave the results

φ		Karpow's table		calculated			
$\frac{3\pi}{16}$	Re W(z)	0.8258	(-1)	0.82578	20982	87	(-1)
	Im W(z)	-0.5756	(-1)	-0.57558	57960	19	(-1)
$\frac{\pi}{4}$	Re W(z)	-0.4807	(-1)	-0.48072	64069	85	(-1)
	Im W(z)	0.80640		0.86039	77157	90	

	$\alpha = 0.5, \varphi = 0, \mu = 1$	$\alpha = 0.5, \varphi = 0, \mu = \pi$
	+.358974065163 − 8	+.358974065163 − 8
	+.349527379238 − 8	+.350484305876 − 8
	+.349763546386 − 8	+.349910380902 − 8
	+.349735431249 − 8	+.349791159824 − 8
	+.349736581414 − 8	+.349757481895 − 8
	+.349736223029 − 8	+.349745672474 − 8
	+.349736226328 − 8	+.349740872213 − 8
	+.349736215746 − 8	+.349738695358 − 8
	+.349736214925 − 8	+.349737620889 − 8
	+.349736214336 − 8	+.349737053352 − 8
x = 18	+.197317529117 − 8	+.197318002481 − 8
	+.223872933446 −14	+.223872933446 −14
	+.220480919303 −14	+.220687634823 −14
	+.220530801864 −14	+.220551048413 −14
	+.220527238824 −14	+.220532264402 −14
	+.220527327900 −14	+.220528625107 −14
	+.220527310647 −14	+.220527723448 −14
	+.220527310747 −14	+.220527457851 −14
	+.220527310541 −14	+.220527368638 −14
	+.220527310530 −14	+.220527335394 −14
	+.220527310526 −14	+.220527321913 −14
x = 32	+.124419211486 −14	+.124419217910 −14
	+.272766419342 −22	+.272766419342 −22
	+.270092238760 −22	+.270200007556 −22
	+.270117952035 −22	+.270125153891 −22
	+.270116739144 −22	+.270117977823 −22
	+.270116759360 −22	+.270116987488 −22
	+.270116756726 −22	+.270116809341 −22
	+.270116756736 −22	+.270116770584 −22
	+.270116756721 −22	+.270116760821 −22
	+.270116756721 −22	+.270116758055 −22
	+.270116756721 −22	+.270116757190 −22
x = 50	+.152397060483 −22	+.152397060748 −22

table 1.

The upper ten numbers are the values of the first 10 partial sums of $\Gamma_{\frac{1}{2}}\left(\frac{x}{2}^{2}\right)$, the last one is the value of $Q(x) = \frac{1}{\sqrt{\pi}}\Gamma_{\frac{1}{2}}\left(\frac{x}{2}^{2}\right)$.

Clearly the convergence for $\mu = 1$ is more rapid than for $\mu = \pi$.

r = 1	r = 17
+.36787 94412	+.24352 57482 − 8
−.18393 97206	−.01352 92083 − 8
+.06131 32402	+.00071 20636 − 8
−.03065 66201	−.00007 12063 − 8
+.01226 26480	+.00000 67815 − 8
−.00715 32113	−.00000 10789 − 8
+.00277 36942	+.00000 01274 − 8
−.00197 07828	−.00000 00302 − 8
+.00060 82663	+.00000 00033 − 8
−.00063 25969	−.00000 00013 − 8

table 2.

Ten terms of $\Gamma_\alpha(z)$, $z = r.\exp(i\varphi)$. Here $\alpha = 0$, $\mu = 1$, $\varphi = 0$.

table 3.

Comparison of the faculty-series with the powerseries and the asymptotic expansion: value of n versus relative accuracy.

r = 1, $\varphi = 0$

n	faculty-series (μ=1) rel.acc.	powerseries rel.acc.	asympt.exp. rel.acc.
0	1.00000		≥ 1.00000
1	1.00000	1.00000	
2	0.25000	0.33333	
3	0.14286	0.06897	
4	0.05405	0.01310	
5	0.03256	0.00209	
6	0.01247	0.00029	
7	0.00894	0.00008	
8	0.00275	0.00001	
9	0.00287	< 0.00001	

r = 5, $\varphi = 0$

n	faculty-series (μ=1) rel.acc.	powerseries rel.acc.	asympt.exp. rel.acc.
0	1.00000		1.00000
1	0.20000	≥ 1.00000	0.25000
2	0.02778		0.09091
3	0.00699		0.05769
4	0.00155		0.04412 (least value)
5	0.00054		0.04615
6	0.00013		0.05248
7	0.00006		0.07929
8	0.00001		0.11259
9	0.00001		0.25416

table 3 (continued).

r = 9, φ = 0

n	faculty-series rel.acc.	powerseries rel.acc.	asympt.exp. rel.acc.
0	1.00000		1.00000
1	0.11111	$\geq$ 1.00000	0.12500
2	0.01000		0.02703
3	0.00167		0.00909
4	0.00026		0.00403
5	0.00006		0.00224
6	0.00001		0.00149
7	0.00000 412		0.00116
8	0.00000 067		0.00103 (least value)
9	0.00000 039		0.00104